Yangians and transvector algebras

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Abstract

Olshanski's centralizer construction provides a realization of the Yangian $\mathrm{Y}(m)$ for the Lie algebra $\mathfrak{gl}(m)$ as a subalgebra in the projective limit algebra $\mathrm{A}_m = \lim\operatorname{proj}\mathrm{A}_m(n)$ as $n \to \infty$, where $\mathrm{A}_m(n)$ is the centralizer of $\mathfrak{gl}(n-m)$ in the enveloping algebra $\mathrm{U}(\mathfrak{gl}(n))$. We give a modified version of this construction based on a quantum analog of Sylvester's theorem. We then use it to get an algebra homomorphism from the Yangian $\mathrm{Y}(n)$ to the transvector algebra associated with the pair $\mathfrak{gl}(m) \subset \mathfrak{gl}(m+n)$. The results are applied to identify the elementary representations of the Yangian by constructing their highest vectors explicitly in terms of elements of the transvector algebra.

0 Introduction

Let $\mathrm{A}_m(n)$ denote the centralizer of $\mathfrak{gl}(n-m)$ in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(n))$. In particular, $\mathrm{A}_0(n)$ is the center of $\mathrm{U}(\mathfrak{gl}(n))$. It was shown by Olshanski [20, 21] that for any fixed m there exists a chain of natural homomorphisms

$$\mathrm{A}_m(m) \leftarrow \mathrm{A}_m(m+1) \leftarrow \cdots \leftarrow \mathrm{A}_m(n) \leftarrow \cdots \tag{0.1}$$

and one can define the corresponding projective limit algebra

$$\mathrm{A}_m = \lim\operatorname{proj} \mathrm{A}_m(n), \qquad n \to \infty. \tag{0.2}$$

The algebra A_0 is isomorphic to an algebra of polynomials in countably many variables while for $m > 0$ one has the tensor product decomposition [20, 21]:

$$\mathrm{A}_m = \mathrm{A}_0 \otimes \mathrm{Y}(m), \tag{0.3}$$

where $\mathrm{Y}(m) = \mathrm{Y}(\mathfrak{gl}(m))$ is the *Yangian* for the Lie algebra $\mathfrak{gl}(m)$; see the definition in Section 1 below. The algebra $\mathrm{Y}(m)$ first appeared in the works of Faddeev's school on the Yang–Baxter equation; see e.g. [11, 12, 23]. The Yangian $\mathrm{Y}(m)$ is a Hopf algebra which can be regarded as a deformation of the enveloping algebra $\mathrm{U}(\mathfrak{gl}(m)[x])$, where $\mathfrak{gl}(m)[x]$ is the Lie algebra of $\mathfrak{gl}(m)$-valued polynomials [5].

The key part of the proof of the decomposition (0.3) is a construction of algebra homomorphisms

$$\mathrm{Y}(m) \to \mathrm{A}_m(n), \qquad n = m, m+1, \ldots \tag{0.4}$$

compatible with the chain (0.1). Then one shows that this defines an embedding

$$\mathrm{Y}(m) \hookrightarrow \mathrm{A}_m \tag{0.5}$$

so that the Yangian $\mathrm{Y}(m)$ can be identified with a subalgebra in A_m; see [20, 21].

Note that a projective limit algebra of type (0.2) can be also constructed for the series of the orthogonal and symplectic Lie algebras. This limit algebra turns out to be related with the corresponding twisted Yangians; see [15, 22] for more details.

In this paper we construct a new family of homomorphisms (0.4) which define a new embedding (0.5). Then we give a modified tensor product decomposition of type (0.3). Our argument is based on a quantum analog of Sylvester's theorem for the Yangians (Theorem 1.3); cf. [8, 10].

We also use Theorem 1.3 to identify the so-called elementary representations of $\mathrm{Y}(n)$. They naturally arise from the centralizer construction as follows. Consider a finite-dimensional irreducible representation $L(\lambda)$ of the Lie algebra $\mathfrak{gl}(m+n)$ with the highest weight λ and denote by $L(\lambda)^+_\mu$ the subspace in $L(\lambda)$ of $\mathfrak{gl}(m)$-highest vectors of weight μ. It is well-known (see e.g. [4, Section 9.1]) that $L(\lambda)^+_\mu$ is an irreducible representation of the centralizer $\mathrm{A} = \mathrm{U}(\mathfrak{gl}(m+n))^{\mathfrak{gl}(m)}$. We can make $L(\lambda)^+_\mu$ into a $\mathrm{Y}(n)$-module via the homomorphism $\mathrm{Y}(n) \to \mathrm{A}$; cf. (0.4). This module can be shown to be irreducible; see Section 4.1.

On the other hand, the subspace $L(\lambda)^+$ of $\mathfrak{gl}(m)$-highest vectors in $L(\lambda)$ is preserved by the so-called *raising* and *lowering operators*. The latter generate the *transvector algebra* Z associated with the pair $\mathfrak{gl}(m) \subset \mathfrak{gl}(m+n)$ (which is sometimes called the *S-algebra* or *Mickelsson algebra*); see [26]–[28]. By definition of Z (see Section 3) there is a natural algebra homomorphism $\mathrm{A} \to \mathrm{Z}$ so that the action of generators of $\mathrm{Y}(n)$ in $L(\lambda)^+_\mu$ can be explicitly expressed in terms of the lowering and raising operators (see Theorem 3.1).

We explicitly construct the highest vector of the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$ in terms of the lowering operators and calculate its highest weight. We also identify $L(\lambda)^+_\mu$ as a $\mathrm{Y}(\mathfrak{sl}(n))$-module by calculating its Drinfeld polynomials. They turn out to be the same as those found by Nazarov and Tarasov [17] for a different representation of $\mathrm{Y}(\mathfrak{sl}(n))$ in $L(\lambda)^+_\mu$ defined via the Olshanski homomorphism (0.4), which shows that these two $\mathrm{Y}(\mathfrak{sl}(n))$-modules are isomorphic.

Following [17] we call the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$ *elementary*. These representations play an important role in the classification of the $\mathrm{Y}(n)$-modules with a semisimple action of the Gelfand–Tsetlin subalgebra; see [3, 17]. In particular, it was proved in [17, Theorem 4.1] that, up to an automorphism of $\mathrm{Y}(n)$, any such module is isomorphic to a tensor product of elementary representations.

1 Quantum Sylvester's theorem

A detailed description of the algebraic structure of the Yangian for the Lie algebra $\mathfrak{gl}(n)$ is given in the expository paper [14]. In this section we reproduce some of those results and use them to prove a quantum analog of Sylvester's theorem.

The *Yangian* $\mathrm{Y}(n) = \mathrm{Y}(\mathfrak{gl}(n))$ is the complex associative algebra with the gener-

ators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$ where $1 \le i, j \le n$, and the defining relations

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u-v}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)), \tag{1.1}$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \dots \in \mathrm{Y}(n)[[u^{-1}]]$$

and u is a formal variable. Introduce the matrix

$$T(u) := \sum_{i,j=1}^{n} t_{ij}(u) \otimes E_{ij} \in \mathrm{Y}(n)[[u^{-1}]] \otimes \mathrm{End}\,\mathbb{C}^n,$$

where the E_{ij} are the standard matrix units. Then the relations (1.1) are equivalent to the single relation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v). \tag{1.2}$$

Here $T_1(u)$ and $T_2(u)$ are regarded as elements of $\mathrm{Y}(n)[[u^{-1}]] \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n$, the subindex of $T(u)$ indicates to which copy of $\mathrm{End}\,\mathbb{C}^n$ this matrix corresponds, and

$$R(u) = 1 - Pu^{-1}, \qquad P = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \in (\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}.$$

The *quantum determinant* $\mathrm{qdet}\,T(u)$ of the matrix $T(u)$ is a formal series in u^{-1} with coefficients from $\mathrm{Y}(n)$ defined by

$$\mathrm{qdet}\,T(u) = \sum_{p \in \mathfrak{S}_n} \mathrm{sgn}(p)\, t_{p(1)1}(u) \cdots t_{p(n)n}(u-n+1). \tag{1.3}$$

The coefficients of the quantum determinant $\mathrm{qdet}\,T(u)$ are algebraically independent generators of the center of the algebra $\mathrm{Y}(n)$.

We shall need a generalization of the relation (1.2) for elements of multiple tensor products of the form $\mathrm{Y}(n)[[u^{-1}]] \otimes \mathrm{End}\,\mathbb{C}^n \otimes \dots \otimes \mathrm{End}\,\mathbb{C}^n$. For an operator $X \in \mathrm{End}\,\mathbb{C}^n$ and a number $s = 1, 2, \dots$ we set

$$X_i := 1^{\otimes(i-1)} \otimes X \otimes 1^{\otimes(s-i)} \in (\mathrm{End}\,\mathbb{C}^n)^{\otimes s}, \quad 1 \le i \le s.$$

Similarly, if $X \in (\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$ then for any i, j such that $1 \le i, j \le s$ and $i \ne j$, we denote by X_{ij} the operator in $(\mathbb{C}^n)^{\otimes s}$ which acts as X in the product of ith and jth copies and as 1 in all other copies. Let $u_1, \dots, u_s$ be formal variables. Set

$$R(u_1, \dots, u_s) := (R_{s-1,s})(R_{s-2,s}R_{s-2,s-1}) \cdots (R_{1s} \cdots R_{12}), \tag{1.4}$$

where we abbreviate $R_{ij} := R_{ij}(u_i - u_j)$. Note that the following (Yang–Baxter) relation is satisfied by the R_{ij}

$$R_{ij}R_{ir}R_{jr} = R_{jr}R_{ir}R_{ij}, \tag{1.5}$$

and the factors R_{ij} and R_{rs} with distinct indices are permutable in (1.4). This allows one to deduce the following identity from (1.2):

$$R(u_1, \ldots, u_s)\, T_1(u_1) \cdots T_s(u_s) = T_s(u_s) \cdots T_1(u_1)\, R(u_1, \ldots, u_s). \tag{1.6}$$

The symmetric group $\mathfrak{S}_s$ naturally acts in the tensor space $(\mathbb{C}^n)^{\otimes s}$ by permutations of the tensor factors. If the variables in (1.4) satisfy the condition $u_i - u_{i+1} = 1$ for $i = 1, \ldots, s-1$ then $R(u_1, \ldots, u_s)$ becomes the antisymmetrization operator in $(\mathbb{C}^n)^{\otimes s}$:

$$R(u_1, \ldots, u_s) = A_s := \sum_{q \in \mathfrak{S}_s} \operatorname{sgn}(q)\, Q, \tag{1.7}$$

where Q is the operator in $(\mathbb{C}^n)^{\otimes s}$ corresponding to a permutation $q \in \mathfrak{S}_s$. By (1.6) we then have

$$A_s\, T_1(u) \cdots T_s(u - s + 1) = T_s(u - s + 1) \cdots T_1(u)\, A_s. \tag{1.8}$$

The operator (1.8) can be written in the form

$$\sum t^{\,a_1 \cdots a_s}_{\,b_1 \cdots b_s}(u) \otimes E_{a_1 b_1} \otimes \cdots \otimes E_{a_s b_s},$$

summed over the indices $a_i, b_i \in \{1, \ldots, n\}$, where $t^{\,a_1 \cdots a_s}_{\,b_1 \cdots b_s}(u) \in \mathrm{Y}(n)[[u^{-1}]]$. In particular, $t^{\,a}_{\,b}(u) = t_{ab}(u)$. Note that by (1.8) the series $t^{\,a_1 \cdots a_s}_{\,b_1 \cdots b_s}(u)$ is antisymmetric with respect to permutations of the upper indices and of the lower indices. It can be given by the following explicit formulas

$$t^{\,a_1 \cdots a_s}_{\,b_1 \cdots b_s}(u) = \sum_{\sigma \in \mathfrak{S}_s} \operatorname{sgn}(\sigma)\, t_{a_{\sigma(1)} b_1}(u) \cdots t_{a_{\sigma(s)} b_s}(u - s + 1) \tag{1.9}$$

$$= \sum_{\sigma \in \mathfrak{S}_s} \operatorname{sgn}(\sigma)\, t_{a_1 b_{\sigma(1)}}(u - s + 1) \cdots t_{a_s b_{\sigma(s)}}(u). \tag{1.10}$$

Note that $t^{\,1 \cdots n}_{\,1 \cdots n}(u) = \operatorname{qdet} T(u)$; see (1.3).

Proposition 1.1 *We have the relations*

$$[t^{a_1\cdots a_k}_{b_1\cdots b_k}(u), t^{c_1\cdots c_l}_{d_1\cdots d_l}(v)] = \sum_{p=1}^{\min\{k,l\}} \frac{(-1)^{p-1}\, p!}{(u-v-k+1)\cdots(u-v-k+p)}$$

$$\sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_p}} \Big(t^{a_1\cdots c_{j_1}\cdots c_{j_p}\cdots a_k}_{b_1\ \cdots\ \ b_k}(u)\, t^{c_1\cdots a_{i_1}\cdots a_{i_p}\cdots c_l}_{d_1\ \cdots\ \ d_l}(v) - t^{c_1\ \cdots\ \ c_l}_{d_1\cdots b_{i_1}\cdots b_{i_p}\cdots d_l}(v)\, t^{a_1\ \cdots\ \ a_k}_{b_1\cdots d_{j_1}\cdots d_{j_p}\cdots b_k}(u) \Big).$$

Here the p-tuples of upper indices $(a_{i_1},\dots,a_{i_p})$ and $(c_{j_1},\dots,c_{j_p})$ are respectively interchanged in the first summand on the right hand side while the p-tuples of lower indices $(b_{i_1},\dots,b_{i_p})$ and $(d_{j_1},\dots,d_{j_p})$ are interchanged in the second summand.

Proof. Let us take $s=k+l$ in (1.6) and specialize the variables u_i as follows:

$$u_i = u-i+1, \quad i=1,\dots,k \quad \text{and} \quad u_{k+j}=v-j+1, \quad j=1,\dots,l. \tag{1.11}$$

Using (1.5) we can bring the product (1.4) to the form

$$R(u_1,\dots,u_{k+l}) = \Bigg(\prod_{j=1,\dots,l}^{\rightarrow} R_{k,k+j}\cdots R_{1,k+j} \Bigg) R(u_1,\dots,u_k)\, R(u_{k+1},\dots,u_{k+l}). \tag{1.12}$$

However, by (1.7) and (1.11)

$$R(u_1,\dots,u_k) = A_k, \qquad R(u_{k+1},\dots,u_{k+l}) = A'_l, \tag{1.13}$$

where A'_l is the antisymmetrizer corresponding to the set of indices $\{k+1,\dots,k+l\}$. Let us show that the expression (1.12) can then be written as

$$\widetilde{R}(u,v)\, A_k A'_l \tag{1.14}$$

with

$$\widetilde{R}(u,v) = \sum_{p=0}^{\min\{k,l\}} \frac{(-1)^p\, p!}{(u-v-k+1)\cdots(u-v-k+p)} \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_p}} P_{i_1,k+j_1}\cdots P_{i_p,k+j_p}. \tag{1.15}$$

Indeed, if indices $i_1,\dots,i_s\in\{1,\dots,k\}$ are distinct then

$$P_{i_1,k+j}\cdots P_{i_s,k+j}\, A_k = (-1)^{s-1} P_{i_s,k+j}\, A_k.$$

This implies that for each j

$$R_{k,k+j}\cdots R_{1,k+j}\, A_k = \left(1 - \frac{P_{1,k+j}+\cdots+P_{k,k+j}}{u-v-k+j}\right) A_k. \tag{1.16}$$

Similarly, if indices $j_1, \dots, j_s \in \{1, \dots, l\}$ are distinct then

$$P_{i,k+j_1} \cdots P_{i,k+j_s} A'_l = (-1)^{s-1} P_{i,k+j_s} A'_l. \tag{1.17}$$

Moreover, if $i_1 \neq i_2$ and $j_1 \neq j_2$ then

$$P_{i_1,k+j_1} P_{i_2,k+j_2} A_k A'_l = P_{i_1,k+j_2} P_{i_2,k+j_1} A_k A'_l.$$

Using this relation together with (1.17) we bring (1.12) to the form (1.14) by a straightforward calculation. Now (1.6) takes the form

$$\begin{aligned} &\widetilde{R}(u,v) A_k T_1(u) \cdots T_k(u-k+1)\, A'_l\, T_{k+1}(v) \cdots T_{k+l}(v-l+1) \\ &\quad = A'_l\, T_{k+1}(v) \cdots T_{k+l}(v-l+1)\, A_k T_1(u) \cdots T_k(u-k+1) \widetilde{R}(u,v), \end{aligned} \tag{1.18}$$

where we have used the identity (1.8) and the fact that $\widetilde{R}(u,v)\, A_k A'_l = A_k A'_l\, \widetilde{R}(u,v)$ which is easy to verify. To complete the proof we take the coefficients at $E_{a_1 b_1} \otimes \cdots \otimes E_{a_k b_k} \otimes E_{c_1 d_1} \otimes \cdots \otimes E_{c_l d_l}$ on both sides of (1.18) (or, equivalently, apply both sides to the basis vector $e_{b_1} \otimes \cdots \otimes e_{b_k} \otimes e_{d_1} \otimes \cdots \otimes e_{d_l}$ of $(\mathbb{C}^n)^{\otimes(k+l)}$ and compare the coefficients at the vector $e_{a_1} \otimes \cdots \otimes e_{a_k} \otimes e_{c_1} \otimes \cdots \otimes e_{c_l}$, where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{C}^n$). □

Introduce the *quantum comatrix* $\widehat{T}(u) = (\widehat{t}_{ij}(u))$ for the matrix $T(u)$ by

$$\widehat{t}_{ij}(u) = (-1)^{i+j} t\,^{1 \cdots \widehat{j} \cdots n}_{1 \cdots \widehat{i} \cdots n}(u),$$

where the hats on the right hand side indicate the indices to be omitted. Equivalently, $\widehat{T}(u)$ can be defined by

$$A_n T_1(u) \cdots T_{n-1}(u-n+2) = A_n \widehat{T}_n(u).$$

Taking $s = n$ in (1.8) we derive the identity

$$\widehat{T}(u) T(u-n+1) = \operatorname{qdet} T(u). \tag{1.19}$$

The Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}(n)$ (see e.g. [14, Corollary 1.23]) implies that for $m \leq n$ the Yangians $\mathrm{Y}(m)$ and $\mathrm{Y}(n-m)$ can be identified with the subalgebras in $\mathrm{Y}(n)$ generated by the coefficients of the series $t_{ij}(u)$ with $i, j \leq m$ and $m < i, j \leq n$, respectively. For any indices $i, j \leq m$ introduce the following series with coefficients in $\mathrm{Y}(n)$

$$\widetilde{t}_{ij}(u) = t\,^{i, m+1 \cdots n}_{j, m+1 \cdots n}(u) \tag{1.20}$$

and combine them into the matrix $\widetilde{T}(u)=(\widetilde{t}_{ij}(u))$. For subsets $\mathcal{P}$ and $\mathcal{Q}$ in $\{1,\dots,n\}$ and an $n\times n$-matrix X we shall denote by $X_{\mathcal{P}\mathcal{Q}}$ the submatrix of X whose rows and columns are enumerated by $\mathcal{P}$ and $\mathcal{Q}$ respectively. Set $\mathcal{A}=\{1,\dots,m\}$ and $\mathcal{B}=\{m+1,\dots,n\}$. We shall need the following generalization of (1.19).

Proposition 1.2 *We have the identity*

$$\widehat{T}(u)_{\mathcal{A}\mathcal{A}}\,\widetilde{T}(u-m+1)=\operatorname{qdet}T(u)\operatorname{qdet}T(u-m+1)_{\mathcal{B}\mathcal{B}}. \tag{1.21}$$

Proof. For $m=1$ the relation (1.21) is trivial. We shall assume that $m\geq 2$. Consider the identity (1.6) with $s=2n-m$ and specialize the variables u_i as follows

$$u_i=u-i+1,\quad i=1,\dots,n-1\quad\text{and}\quad u_{n+j-1}=v-j+1,\quad j=1,\dots,n-m+1.$$

We shall show that (1.6) then becomes a rational function in v with a simple pole at $v=u-m+1$ and calculate the corresponding residue in two different ways. First, write (1.6) in the form (1.18) with $k=n-1$ and $l=n-m+1$. Then fix indices $i,j\in\{1,\dots,m\}$ and apply the left hand side of (1.18) to the basis vector

$$e_1\otimes\dots\otimes e_{i-1}\otimes e_{i+1}\otimes\dots\otimes e_n\otimes e_j\otimes e_{m+1}\otimes\dots\otimes e_n \tag{1.22}$$

in $(\mathbb{C}^n)^{\otimes(2n-m)}$ and take the coefficient at the vector

$$e_1\otimes\dots\otimes e_{i-1}\otimes e_{i+1}\otimes\dots\otimes e_n\otimes e_i\otimes e_{m+1}\otimes\dots\otimes e_n. \tag{1.23}$$

A straightforward calculation shows that this matrix element equals

$$\frac{u-v-n+1}{u-v-m+1}\left(\widehat{t}_{ii}(u)\widetilde{t}_{ij}(v)+\frac{1}{u-v-m+2}\sum_{a=1,\,a\neq i}^{m}\widehat{t}_{ia}(u)\widetilde{t}_{aj}(v)\right).$$

Multiplying this expression by $u-v-m+1$ and setting $v=u-m+1$ we obtain

$$(m-n)\left(\widehat{T}(u)_{\mathcal{A}\mathcal{A}}\,\widetilde{T}(u-m+1)\right)_{ij}.$$

On the other hand, transform $R(u_1,\dots,u_{2n-m})$ applying (1.12) and (1.13) with $k=n-1$ and $l=n-m+1$. Using (1.16) we can write $R(u_1,\dots,u_{2n-m})$ as

$$\left(\prod_{j=1,\dots,l-1}^{\rightarrow}R_{k,k+j}\cdots R_{1,k+j}\right)\left(1-\frac{P_{1,k+l}+\dots+P_{k,k+l}}{u-v-k+l}\right)A_kA'_l. \tag{1.24}$$

Note that $(l-1)!\, A'_l = A'_{l-1} A'_l$. Therefore, repeating the argument of the proof of Proposition 1.1 we can transform (1.24) to get the expression

$$\frac{1}{(l-1)!}\, \widetilde{R}(u,v) A_k A'_{l-1} \left(1 - \frac{P_{1,k+l} + \cdots + P_{k,k+l}}{u - v - k + l}\right) A'_l, \tag{1.25}$$

where $\widetilde{R}(u,v)$ is given by (1.15) with l replaced by $l-1$. Finally, put $k = n-1$, $l = n-m+1$ into (1.25), multiply it by $u-v-m+1$ and then put $v = u-m+1$. This gives

$$(m-n) \sum P_{i_1,n} \cdots P_{i_{n-m},2n-m-1} (1 - P_{1,2n-m} - \cdots - P_{n-1,2n-m}) A_{n-1} A'_{n-m+1}, \tag{1.26}$$

summed over the sets of indices $1 \le i_1 < \cdots < i_{n-m} \le n-1$. Note that

$$\widetilde{A}_n := (1 - P_{1,2n-m} - \cdots - P_{n-1,2n-m}) A_{n-1}$$

is the antisymmetrizer corresponding to the subset of indices $\{1, \ldots, n-1, 2n-m\}$. The application of the operator (1.26) to the vector (1.22) gives the zero vector unless $i = j$ since it is annihilated by $\widetilde{A}_n A'_{n-m+1}$. Suppose now that $i = j$. By (1.8) we can write

$$A'_{n-m+1} T_n(u-m+1) \cdots T_{2n-m}(u-n+1) = \\ T_{2n-m}(u-n+1) \cdots T_n(u-m+1) A'_{n-m+1}.$$

The residue of the left hand side of (1.6) at $v = u-m+1$ then takes the form

$$(m-n) \sum P_{i_1,n} \cdots P_{i_{n-m},2n-m-1} \widetilde{A}_n T_1(u) \cdots T_{n-1}(u-n+2) T_{2n-m}(u-n+1) \\ T_{2n-m-1}(u-n+2) \cdots T_n(u-m+1) A'_{n-m+1}.$$

The diagonal matrix element of this operator corresponding to the vector (1.23) equals

$$(m-n)\, \mathrm{qdet}\, T(u)\, \mathrm{qdet}\, T(u-m+1)_{\mathcal{B}\mathcal{B}}$$

which completes the proof. ◻

We are now in a position to prove a quantum analog of Sylvester's theorem for the Yangians (Theorem 1.3). An analog of Sylvester's theorem for the quantized algebra of functions on $GL(n)$ was given by Krob and Leclerc [10] with the use of the quasi-determinant version of this theorem due to Gelfand and Retakh [8]. As was remarked in [10], the same approach can be used for the case of the Yangians. We give a different proof based on Propositions 1.1 and 1.2. We use notation (1.20).

Theorem 1.3 *The mapping*

$$t_{ij}(u) \mapsto \widetilde{t}_{ij}(u), \qquad 1 \le i, j \le m \tag{1.27}$$

defines an algebra homomorphism $\mathrm{Y}(m) \to \mathrm{Y}(n)$*. Moreover, one has the identity*

$$\operatorname{qdet} \widetilde{T}(u) = \operatorname{qdet} T(u) \operatorname{qdet} T(u-1)_{\mathcal{B}\mathcal{B}} \cdots \operatorname{qdet} T(u-m+1)_{\mathcal{B}\mathcal{B}}. \tag{1.28}$$

Proof. To check that the elements (1.20) satisfy the defining relations (1.1) we use Proposition 1.1. This proves the first part of the theorem.

To prove (1.28) we use induction on m. For $m = 1$ the identity is trivial. Let $m \ge 2$. We obtain from (1.19) that

$$\widehat{T}(u) = \operatorname{qdet} T(u)\, T(u-n+1)^{-1}.$$

Taking the mmth entry on both sides we obtain

$$\operatorname{qdet} T(u)_{\mathcal{C}\mathcal{C}} = \operatorname{qdet} T(u) \left(T(u-n+1)^{-1}\right)_{mm}, \tag{1.29}$$

where $\mathcal{C} = \{1, \dots, m-1, m+1, \dots, n\}$. Similarly, using the homomorphism (1.27) we can write the corresponding analog of (1.19) for the matrix $\widetilde{T}(u)$ which gives

$$\operatorname{qdet} \widetilde{T}(u)_{\mathcal{A}'\mathcal{A}'} = \operatorname{qdet} \widetilde{T}(u) \left(\widetilde{T}(u-m+1)^{-1}\right)_{mm}, \tag{1.30}$$

where $\mathcal{A}' = \{1, \dots, m-1\}$. By the induction hypothesis,

$$\operatorname{qdet} \widetilde{T}(u)_{\mathcal{A}'\mathcal{A}'} = \operatorname{qdet} T(u)_{\mathcal{C}\mathcal{C}} \operatorname{qdet} T(u-1)_{\mathcal{B}\mathcal{B}} \cdots \operatorname{qdet} T(u-m+2)_{\mathcal{B}\mathcal{B}}.$$

It remains to apply (1.29), (1.30) and the identity

$$\left(T(u-n+1)^{-1}\right)_{\mathcal{A}\mathcal{A}} = \widetilde{T}(u-m+1)^{-1} \operatorname{qdet} T(u-m+1)_{\mathcal{B}\mathcal{B}},$$

which is a corollary of (1.19) and (1.21). □

2 Modified centralizer construction

We start with the definition of the Olshanski algebra A_m; see [20, 21]. Fix a nonnegative integer m and for any $n \ge m$ denote by $\mathfrak{g}_m(n)$ the subalgebra in the general linear Lie algebra $\mathfrak{gl}(n)$ spanned by the basis elements E_{ij} with $m+1 \le i, j \le n$.

The subalgebra $\mathfrak{g}_m(n)$ is isomorphic to $\mathfrak{gl}(n-m)$. Let $\mathrm{A}_m(n)$ denote the centralizer of $\mathfrak{g}_m(n)$ in the universal enveloping algebra $\mathrm{A}(n) := \mathrm{U}(\mathfrak{gl}(n))$. In particular, $\mathrm{A}_0(n)$ is the center of $\mathrm{U}(\mathfrak{gl}(n))$. Let $\mathrm{A}(n)^0$ denote the centralizer of E_{nn} in $\mathrm{A}(n)$ and let $\mathrm{I}(n)$ be the left ideal in $\mathrm{A}(n)$ generated by the elements E_{in}, $i = 1, \dots, n$. Then $\mathrm{I}(n)^0 := \mathrm{I}(n) \cap \mathrm{A}(n)^0$ is a two-sided ideal in $\mathrm{A}(n)^0$ and one has a vector space decomposition

$$\mathrm{A}(n)^0 = \mathrm{I}(n)^0 \oplus \mathrm{A}(n-1).$$

Therefore the projection of $\mathrm{A}(n)^0$ onto $\mathrm{A}(n-1)$ with the kernel $\mathrm{I}(n)^0$ is an algebra homomorphism. Its restriction to the subalgebra $\mathrm{A}_m(n)$ defines a filtration preserving homomorphism

$$\pi_n : \mathrm{A}_m(n) \to \mathrm{A}_m(n-1) \tag{2.1}$$

so that one can define the algebra A_m as the projective limit (0.2) in the category of filtered algebras. In other words, an element of the algebra A_m is a sequence of the form $a = (a_m, a_{m+1}, \dots, a_n, \dots)$ where $a_n \in \mathrm{A}_m(n)$, $\pi_n(a_n) = a_{n-1}$ for $n > m$, and

$$\deg a := \sup_{n \geq m} \deg a_n < \infty,$$

where $\deg a_n$ denotes the degree of a_n in the universal enveloping algebra $\mathrm{A}(n)$.

The algebra A_0 is isomorphic to the algebra of virtual Laplace operators for the Lie algebra $\mathfrak{gl}(\infty)$ [21]. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be an integer sequence such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_i = 0$ for all sufficiently large i and let L be a representation of $\mathfrak{gl}(\infty)$ with the highest weight λ. Then every element $a \in \mathrm{A}_0$ acts in L as a scalar. A family of algebraically independent generators of A_0 was constructed in [21, Remark 2.1.20] (see also [9]) and their eigenvalues in L were calculated. We shall need a different family of generators of A_0 which will be more suitable for our purposes.

Let $E = (E_{ij})$ denote the infinite matrix whose ijth entry is the basis element E_{ij} of $\mathfrak{gl}(\infty)$ and let $E^{(n)}$ be its submatrix corresponding to the subset of indices $1 \leq i, j \leq n$. It is well-known and can be easily verified that the mapping

$$T(u) \mapsto 1 + E^{(n)} u^{-1} \tag{2.2}$$

defines an algebra epimorphism $\mathrm{Y}(n) \to \mathrm{A}(n)$. Consider the quantum determinant

$$\mathrm{qdet}\,(1 + E^{(n)} u^{-1}) = \sum_{p \in \mathfrak{S}_n} \mathrm{sgn}(p)\, (1 + E u^{-1})_{p(1)1} \cdots (1 + E(u-n+1)^{-1})_{p(n)n}$$

and write

$$\operatorname{qdet}(1+E^{(n)}u^{-1}) = 1+\mathcal{E}_1^{(n)}u^{-1}+\mathcal{E}_2^{(n)}u^{-2}+\cdots.$$

Since the coefficients of the series $\operatorname{qdet} T(u)$ are central in $\mathrm{Y}(n)$, the elements $\mathcal{E}_i^{(n)}$ belong to the center $\mathrm{A}_0(n)$ of $\mathrm{A}(n)$. Further, by definition (2.1) we obviously have

$$\pi_n : \operatorname{qdet}(1+E^{(n)}u^{-1}) \mapsto \operatorname{qdet}(1+E^{(n-1)}u^{-1}).$$

So, we may define the elements $\mathcal{E}_i \in \mathrm{A}_0$, $i=1,2,\dots$ as sequences $\mathcal{E}_i = (\mathcal{E}_i^{(n)} | n \geq 1)$. We call the corresponding generating series

$$\operatorname{qdet}(1+Eu^{-1}) := 1+\mathcal{E}_1u^{-1}+\mathcal{E}_2u^{-2}+\cdots \tag{2.3}$$

the *virtual quantum determinant*. This is a particular case of a more general construction of quantum immanants given in [18, 19]. The following proposition is immediate.

Proposition 2.1 *The elements $\mathcal{E}_i$ with $i=1,2,\dots$ are algebraically independent generators of the algebra A_0. Their eigenvalues in a representation L of the Lie algebra $\mathfrak{gl}(\infty)$ with the highest weight λ are given by the formula*

$$\operatorname{qdet}(1+Eu^{-1})|_L = \prod_{i=1}^{\infty} \frac{u+\lambda_i-i+1}{u-i+1}.$$

Consider the homomorphism $\mathrm{Y}(m) \to \mathrm{Y}(n)$ provided by Theorem 1.3 and take its composition with (2.2). We obtain an algebra homomorphism $\Phi_n : \mathrm{Y}(m) \to \mathrm{A}(n)$ given by

$$\Phi_n : t_{ij}(u) \mapsto \operatorname{qdet}(1+Eu^{-1})_{\mathcal{B}_i\mathcal{B}_j}, \tag{2.4}$$

where $\mathcal{B}_i$ denotes the set $\{i, m+1, \dots, n\}$. In other words, $\Phi_n(t_{ij}(u))$ coincides with the image of the element $\widetilde{t}_{ij}(u)$ under (2.2). Proposition 1.1 implies that this image commutes with the elements of the subalgebra $\mathfrak{g}_m(n)$ and so, (2.4) defines a homomorphism $\Phi_n : \mathrm{Y}(m) \to \mathrm{A}_m(n)$. Furthermore, the family of homomorphisms $\{\Phi_n | n \geq m\}$ is obviously compatible with the homomorphisms (2.1) and thus defines an algebra homomorphism $\Phi : \mathrm{Y}(m) \to \mathrm{A}_m$. Introducing the virtual quantum determinants of the matrices $(1+Eu^{-1})_{\mathcal{B}_i\mathcal{B}_j}$ with $\mathcal{B}_i = \{i, m+1, m+2, \dots\}$ as in (2.3), we can represent this homomorphism as follows

$$\Phi : t_{ij}(u) \mapsto \operatorname{qdet}(1+Eu^{-1})_{\mathcal{B}_i\mathcal{B}_j}.$$

The following is a modified version of the Olshanski theorem; see [20, 21]. Denote by $\widetilde{\mathrm{A}}_0$ the commutative algebra generated by the coefficients of the virtual quantum determinant $\operatorname{qdet}(1+Eu^{-1})_{\mathcal{B}\mathcal{B}}$ with $\mathcal{B} = \{m+1, m+2, \dots\}$.

Theorem 2.2 *The homomorphism* $\Phi : \mathrm{Y}(m) \to \mathrm{A}_m$ *is injective. Moreover, one has an isomorphism*

$$\mathrm{A}_m = \widetilde{\mathrm{A}}_0 \otimes \mathrm{Y}(m), \tag{2.5}$$

where $\mathrm{Y}(m)$ *is identified with its image under the embedding* Φ.

Proof. Consider the canonical filtration of the universal enveloping algebra $\mathrm{A}(n)$. The corresponding graded algebra $\mathrm{gr}\,\mathrm{A}(n)$ is isomorphic to the symmetric algebra $\mathrm{P}(n)$ of the space $\mathfrak{gl}(n)$. Elements of $\mathrm{P}(n)$ can be naturally identified with polynomials in matrix elements of an $n \times n$-matrix $x = (x_{ij})$. Denote by $\mathrm{P}_m(n)$ the subalgebra of the elements of $\mathrm{P}(n)$ which are invariant under the adjoint action of the Lie algebra $\mathfrak{g}_m(n)$. One can show (see [21]) that the graded algebra $\mathrm{gr}\,\mathrm{A}_m$ is naturally isomorphic to the projective limit P_m of the commutative algebras $\mathrm{P}_m(n)$ with respect to homomorphisms $\mathrm{P}_m(n) \to \mathrm{P}_m(n-1)$ analogous to (2.1). In particular, we can define the virtual determinants

$$\det(1 + xu^{-1})_{\mathcal{B}_i\mathcal{B}_j}, \qquad \det(1 + xu^{-1})_{\mathcal{B}\mathcal{B}} \tag{2.6}$$

of the submatrices of the infinite matrix $1 + xu^{-1}$, where $x = (x_{ij})_{i,j=1}^{\infty}$, as formal series in u^{-1} with coefficients from P_m. These coefficients coincide with the images in P_m of the coefficients of the virtual quantum determinants of the corresponding submatrices of E. Therefore, both claims of the theorem will follow from the fact that the coefficients of the series (2.6) where $1 \le i, j \le m$ are algebraically independent generators of the algebra P_m. To see this, for a finite value of n consider the sets $\mathcal{B} = \{m+1, m+2, \dots, n\}$ and $\mathcal{B}_i = \{i, m+1, m+2, \dots, n\}$. We have an identity

$$\det(1 + xu^{-1})_{\mathcal{B}_i\mathcal{B}_j} = \det(1 + xu^{-1})_{\mathcal{B}\mathcal{B}} \left|(1 + xu^{-1})_{\mathcal{B}_i\mathcal{B}_j}\right|_{ij}$$

with

$$\left|(1 + xu^{-1})_{\mathcal{B}_i\mathcal{B}_j}\right|_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} (-1)^{k-1} \Lambda_{ij}^{(k)} u^{-k},$$

where $\Lambda_{ij}^{(k)} = \sum x_{ia_1} x_{a_1a_2} \cdots x_{a_{k-1}j}$, summed over the indices $a_r \in \{m+1, \dots, n\}$; see e.g. [7, Section 7.4]. However, the coefficients of the polynomial $\det(1 + xu^{-1})_{\mathcal{B}\mathcal{B}}$ generate the full set of invariants of the matrix $x_{\mathcal{B}\mathcal{B}}$. Further, these coefficients and the elements $\Lambda_{ij}^{(k)}$ generate the algebra $\mathrm{P}_m(n)$ which follows from [21, Section 2.1.10] and [7, Section 7.4]. Finally, given any positive integer M, the coefficients at $u^{-1}, \dots, u^{-M}$ of

$\det(1+xu^{-1})_{\mathcal{BB}}$ and the elements $\Lambda_{ij}^{(k)}$ with $k=1,\dots,M$ and $i,j=1,\dots,m$ are algebraically independent for a sufficiently large n which is implied by [21, Lemma 2.1.11].

$\square$

Remark. For $m>1$ the algebra $\widetilde{\mathrm{A}}_0$ in the decomposition (2.5) can be replaced by A_0. This can be deduced from the classical Sylvester's identity (cf. Theorem 1.3):

$$\det X = \det(1+xu^{-1})\left(\det(1+xu^{-1})_{\mathcal{BB}}\right)^{m-1},$$

where $X=(X_{ij})$ with $X_{ij}=\det(1+xu^{-1})_{\mathcal{B}_i\mathcal{B}_j}$. $\square$

3 Extremal projection and transvector algebras

In the last two sections we shall consider the pair of Lie algebras $\mathfrak{gl}(m)\subset\mathfrak{gl}(m+n)$ with the parameters m and n fixed, where $\mathfrak{gl}(m)$ is spanned by the basis elements E_{ij} of $\mathfrak{gl}(m+n)$ with $i,j=1,\dots,m$.

Denote by $\mathfrak{h}$ the diagonal Cartan subalgebra of $\mathfrak{gl}(m)$ spanned by E_{ii} with $i=1,\dots,m$. Consider the extension of the universal enveloping algebra $\mathrm{A}(m+n)=\mathrm{U}(\mathfrak{gl}(m+n))$

$$\mathrm{A}'(m+n)=\mathrm{A}(m+n)\otimes_{\mathrm{U}(\mathfrak{h})}\mathrm{R}(\mathfrak{h}),$$

where $\mathrm{R}(\mathfrak{h})$ is the field of fractions of the commutative algebra $\mathrm{U}(\mathfrak{h})$. Let J denote the left ideal in $\mathrm{A}'(m+n)$ generated by the elements E_{ij} with $1\le i<j\le m$. The *transvector algebra* $\mathrm{Z}=\mathrm{Z}(\mathfrak{gl}(m+n),\mathfrak{gl}(m))$ is the quotient algebra of the normalizer

$$\operatorname{Norm}\mathrm{J}=\{x\in\mathrm{A}'(m+n)\mid \mathrm{J}x\subseteq\mathrm{J}\}$$

modulo the two-sided ideal J; see [27]. It is an algebra over $\mathbb{C}$ and an $\mathrm{R}(\mathfrak{h})$-bimodule. Generators of Z can be constructed by using the *extremal projection* p for the Lie algebra $\mathfrak{gl}(m)$ [1, 26]. The projection p is an element of an algebra F of formal series and can be defined as follows. For any pair (i,j) such that $1\le i<j\le m$ set

$$p_{ij}=\sum_{k=0}^{\infty}(E_{ji})^k(E_{ij})^k\,\frac{(-1)^k}{k!\,(h_i-h_j+1)\cdots(h_i-h_j+k)},$$

where $h_i := E_{ii} - i + 1$. Then p is given by

$$p = \prod_{i<j} p_{ij},$$

where the product is taken in any *normal ordering* on the pairs (i,j). The positive roots of $\mathfrak{gl}(m)$ with respect to $\mathfrak{h}$ (with the standard choice of the positive root system) are naturally enumerated by the pairs (i,j) and normality means that any composite root lies between its components. The element p is, up to a factor from $\mathrm{R}(\mathfrak{h})$, a unique element of F such that

$$E_{ij}\, p = pE_{ji} = 0 \qquad \text{for} \quad 1 \leq i < j \leq m. \tag{3.1}$$

In particular, p does not depend on the normal ordering and satisfies the conditions $p^2 = p$ and $p^* = p$, where $x \mapsto x^*$ is an involutive anti-automorphism of the algebra F such that $(E_{ij})^* = E_{ji}$.

The action of p on elements of the quotient $\mathrm{A}'(m+n)/\mathrm{J}$ is well-defined and the transvector algebra Z can be identified with the image of $\mathrm{A}'(m+n)/\mathrm{J}$ with respect to the projection p:

$$\mathrm{Z} = p\,(\mathrm{A}'(m+n)/\mathrm{J})\,.$$

An analog of the Poincaré–Birkhoff–Witt theorem holds for the algebra Z [27, 28] so that ordered monomials in the elements E_{ab}, pE_{ia}, pE_{ai}, where $i = 1, \dots, m$ and $a, b = m+1, \dots, m+n$ form a basis of Z as a left or right $\mathrm{R}(\mathfrak{h})$-module. It will be convenient to use the following generators of Z: for $i = 1, \dots, m$ and $a = m+1, \dots, m+n$

$$\begin{aligned} s_{ia} &= pE_{ia}(h_i - h_1) \cdots (h_i - h_{i-1}), \\ s_{ai} &= pE_{ai}(h_i - h_{i+1}) \cdots (h_i - h_m). \end{aligned} \tag{3.2}$$

Explicitly,

$$\begin{aligned} s_{ia} &= \sum_{i>i_1>\cdots>i_s\geq 1} E_{ii_1}E_{i_1i_2}\cdots E_{i_{s-1}i_s}E_{i_sa}(h_i - h_{j_1})\cdots(h_i - h_{j_r}), \\ s_{ai} &= \sum_{i<i_1<\cdots<i_s\leq m} E_{i_1i}E_{i_2i_1}\cdots E_{i_si_{s-1}}E_{ai_s}(h_i - h_{j_1})\cdots(h_i - h_{j_r}), \end{aligned} \tag{3.3}$$

where $s = 0, 1, \dots$ and $\{j_1, \dots, j_r\}$ is the complementary subset to $\{i_1, \dots, i_s\}$ respectively in the set $\{1, \dots, i-1\}$ or $\{i+1, \dots, m\}$. The elements E_{ab}, $a, b =$

$m+1,\dots,m+n$ generate a subalgebra in Z isomorphic to $\mathrm{A}(n)$. Moreover, one also has the following relations in Z which are obtained by using the methods of [27, 28]. Let the indices i,j and a,b,c run through the sets $\{1,\dots,m\}$ and $\{m+1,\dots,m+n\}$, respectively. We have

$$[E_{ab}, s_{ci}] = \delta_{bc} s_{ai}, \qquad [E_{ab}, s_{ic}] = -\delta_{ac} s_{ib}. \tag{3.4}$$

Moreover, if $i \neq j$ then

$$s_{ai} s_{bj} = s_{bj} s_{ai} \, \frac{h_i - h_j + 1}{h_i - h_j} - s_{bi} s_{aj} \, \frac{1}{h_i - h_j},$$

while

$$s_{ai} s_{bi} = s_{bi} s_{ai}. \tag{3.5}$$

Finally, $s_{ia} s_{bj} = s_{bj} s_{ia}$ for $i \neq j$ while

$$s_{ia} s_{bi} = (\delta_{ab}(E_{ii}+m-i)-E_{ab}) \prod_{j=1,\, j\neq i}^{m} (h_i-h_j-1) + \sum_{j=1}^{m} s_{bj} s_{ja} \prod_{k=1,\, k\neq j}^{m} \frac{h_i - h_k - 1}{h_j - h_k}. \tag{3.6}$$

The anti-involution $x \mapsto x^*$ extends to the algebra Z so that $(pE_{ai})^* = pE_{ia}$ [27, 28]. Therefore,

$$(s_{ai})^* = s_{ia} \frac{(h_i - h_{i+1} + 1) \cdots (h_i - h_m + 1)}{(h_i - h_1) \cdots (h_i - h_{i-1})}. \tag{3.7}$$

The centralizer $\mathrm{A} = \mathrm{U}(\mathfrak{gl}(m+n))^{\mathfrak{gl}(m)}$ is a subalgebra in the normalizer Norm J and so, one has the natural algebra homomorphism $\nu : \mathrm{A} \to \mathrm{Z}$. By the results of Section 2 we have an algebra homomorphism

$$\phi : \mathrm{Y}(n) \to \mathrm{A}, \qquad t_{ab}(u) \mapsto \mathrm{qdet}\,(1 + Eu^{-1})_{\mathcal{C}_a \mathcal{C}_b}, \tag{3.8}$$

where $\mathcal{C}_a = \{1,\dots,m,m+a\}$ for $a = 1,\dots,n$; see (2.4). The composition $\psi = \nu \circ \phi$ is an algebra homomorphism $\psi : \mathrm{Y}(n) \to \mathrm{Z}$. The next theorem provides explicit formulas for the images of the generators of $\mathrm{Y}(n)$ with respect to ψ. Note that by (3.8) the image of

$$T_{m+a,m+b}(u) := u(u-1)\cdots(u-m)\, t_{ab}(u) \tag{3.9}$$

under ψ is a polynomial in u.

Theorem 3.1 *We have for $a, b = m+1, \dots, m+n$:*

$$\psi : T_{ab}(u) \mapsto (\delta_{ab}(u-m) + E_{ab}) \prod_{i=1}^{m} (u+h_i) - \sum_{i=1}^{m} s_{ai} s_{ib} \prod_{j=1,\, j\neq i}^{m} \frac{u+h_j}{h_i - h_j} \tag{3.10}$$

and

$$\psi : T_{ab}(u) \mapsto (\delta_{ab} u + E_{ab}) \prod_{i=1}^{m} (u+h_i-1) - \sum_{i=1}^{m} s_{ib} s_{ai} \prod_{j=1,\, j\neq i}^{m} \frac{u+h_j-1}{h_i - h_j}. \tag{3.11}$$

Proof. By (1.9) the image of $T_{ab}(u)$ under the homomorphism (3.8) is

$$\sum_{\sigma} \operatorname{sgn}(\sigma)\, (u+E)_{\sigma(1),1} \cdots (u+E-m+1)_{\sigma(m),m} (u+E-m)_{\sigma(a),b}, \tag{3.12}$$

summed over permutations σ of the set $\{1, \dots, m, a\}$. To find the image of the polynomial (3.12) in Z we shall regard it modulo the ideal J and apply the projection p. Consider first the sum in (3.12) over the permutations σ such that $\sigma(a) = a$. Using (3.1) we obtain that its image in Z$[u]$ is

$$(u+h_1) \cdots (u+h_m)(\delta_{ab}(u-m) + E_{ab}). \tag{3.13}$$

Further, consider the remaining summands in (3.12) and suppose that $\sigma(a) = i$, $\sigma(j) = a$ for certain $i, j \in \{1, \dots, m\}$. Using the property (3.1) of the projection p we find that the image of such a summand in Z$[u]$ can be nonzero only if $i \geq j$ and σ is a cyclic permutation of the form $\sigma = (a, i_1, \dots, i_s)$ where $i = i_1 > i_2 > \cdots > i_s = j$. We have the equality modulo J:

$$E_{ai_s} E_{i_s i_{s-1}} \cdots E_{i_2 i_1} E_{i_1 b} = E_{aj} E_{jb}.$$

Since $\operatorname{sgn}(\sigma) = (-1)^s$ the image in Z$[u]$ of the summand corresponding to σ is

$$(-1)^s\, pE_{aj} E_{jb} (u+h_1) \cdots (\widehat{u+h_{i_s}}) \cdots (\widehat{u+h_{i_1}}) \cdots (u+h_m),$$

where the hats indicate the factors to be omitted. Taking the sum over cycles σ we find that the polynomial $S_{ab}(u) := \psi(T_{ab}(u))$ is given by

$$\begin{aligned} S_{ab}(u) = {} & (\delta_{ab}(u-m) + E_{ab})(u+h_1) \cdots (u+h_m) \\ & - \sum_{j=1}^{m} pE_{aj} E_{jb} (u+h_1) \cdots (u+h_{j-1})(u+h_{j+1}-1) \cdots (u+h_m-1). \end{aligned}$$

Consider the value of $S_{ab}(u)$ at $u = -h_i$ for $i \in \{1, \dots, m\}$ (to make this evaluation well-defined we agree to write the coefficients of the polynomial to the left of the powers of u). We obtain

$$S_{ab}(-h_i) = -\sum_{j=1}^{i} pE_{aj}E_{jb}(h_1 - h_i)\cdots(h_{j-1} - h_i)(h_{j+1} - h_i - 1)\cdots(h_m - h_i - 1).$$

On the other hand, by (3.2) we have

$$s_{ai}s_{ib} = pE_{ai}\, s_{ib}(h_i - h_{i+1} + 1)\cdots(h_i - h_m + 1).$$

Finally, using (3.3) and (3.1) we verify the identity $s_{ai}s_{ib} = (-1)^m S_{ab}(-h_i)$. The difference of $S_{ab}(u)$ and the product (3.13) is a polynomial in u of degree $< m$. So, (3.10) follows from the Lagrange interpolation formula.

The relation (3.11) follows from (3.6) and (3.10); it suffices to consider the values of the polynomial $S_{ab}(u)$ at $u = -h_i + 1$ for $i = 1, \dots, m$. It can also be proved by the above argument where (1.9) is applied to $t^{\,a,1\cdots m}_{\,b,1\cdots m}(u)$ (thus giving another derivation of (3.6)). $\square$

4 Elementary representations of the Yangian

Here we use the results of the previous section to study a special class of representations of the Yangian $\mathrm{Y}(n)$ called elementary. They play an important role in the classification of the representations of $\mathrm{Y}(n)$ with a semisimple action of the Gelfand–Tsetlin subalgebra; see [17].

4.1 Yangian action on the multiplicity space

A representation L of the Yangian $\mathrm{Y}(n)$ is called *highest weight* if it is generated by a nonzero vector ζ (the *highest vector*) such that

$$\begin{aligned} t_{aa}(u)\zeta &= \lambda_a(u)\zeta \qquad &&\text{for} \quad a = 1, \dots, n, \\ t_{ab}(u)\zeta &= 0 \qquad &&\text{for} \quad 1 \le a < b \le n \end{aligned}$$

for certain formal series $\lambda_a(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. The set $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$ is called the *highest weight* of L; cf. [2, 6]. Every finite-dimensional irreducible

representation of the Yangian $\mathrm{Y}(n)$ is highest weight. It contains a unique, up to scalar multiples, highest vector. An irreducible representation of $\mathrm{Y}(n)$ with the highest weight $\lambda(u)$ is finite-dimensional if and only if there exist monic polynomials $P_1(u), \dots, P_{n-1}(u)$ in u (called the *Drinfeld polynomials*) such that

$$\frac{\lambda_a(u)}{\lambda_{a+1}(u)} = \frac{P_a(u+1)}{P_a(u)}, \qquad a = 1, \dots, n-1. \tag{4.1}$$

These results are contained in [6]; see also [2, 13].

Let $\lambda = (\lambda_1, \dots, \lambda_{m+n})$ be an $(m+n)$-tuple of complex numbers satisfying the condition $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for $i = 1, \dots, m+n-1$. Denote by $L(\lambda)$ the irreducible finite-dimensional representation of the Lie algebra $\mathfrak{gl}(m+n)$ with the highest weight λ. It contains a unique nonzero vector ξ (the highest vector) such that

$$\begin{aligned} E_{ii}\xi &= \lambda_i \xi \qquad \text{for} \quad i = 1, \dots, m+n, \\ E_{ij}\xi &= 0 \qquad \text{for} \quad 1 \leq i < j \leq m+n. \end{aligned}$$

Given a $\mathfrak{gl}(m)$-highest weight $\mu = (\mu_1, \dots, \mu_m)$ we denote by $L(\lambda)^+_\mu$ the subspace of $\mathfrak{gl}(m)$-highest vectors in $L(\lambda)$ of weight μ:

$$\begin{aligned} L(\lambda)^+_\mu = \{\eta \in L(\lambda) \mid E_{ii}\eta = \mu_i \eta \qquad &\text{for} \quad i = 1, \dots, m, \\ E_{ij}\eta = 0 \qquad &\text{for} \quad 1 \leq i < j \leq m\}. \end{aligned}$$

The dimension of $L(\lambda)^+_\mu$ coincides with the multiplicity of the $\mathfrak{gl}(m)$-module $L(\mu)$ in the restriction of $L(\lambda)$ to $\mathfrak{gl}(m)$. The multiplicity space $L(\lambda)^+_\mu$ admits a natural structure of an irreducible representation of the centralizer algebra A [4, Section 9.1]. On the other hand, the homomorphism (3.8) allows us to regard $L(\lambda)^+_\mu$ as a $\mathrm{Y}(n)$-module. As it follows from the proof of Theorem 2.2, the algebra A is generated by the image of ϕ and the center of $\mathrm{U}(\mathfrak{gl}(m))$. The elements of the center act in $L(\lambda)^+_\mu$ as scalar operators and so, the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$ is irreducible. Following [17] we call it *elementary*.

We shall use a *contravariant bilinear form* $\langle\, ,\rangle$ on the space $L(\lambda)$ which is uniquely determined by the conditions

$$\langle \xi, \xi \rangle = 1; \qquad \langle Xu, v \rangle = \langle u, X^* v \rangle \quad \text{for} \quad u, v \in L(\lambda), \quad X \in \mathrm{A}(m+n),$$

where $X \mapsto X^*$ is the anti-involution on $\mathrm{A}(m+n)$ such that $(E_{ab})^* = E_{ba}$. The form $\langle\, ,\rangle$ is nondegenerate on $L(\lambda)$, and so is its restriction to each nonzero subspace $L(\lambda)^+_\mu$. Indeed, if there is a vector $\eta \in L(\lambda)^+_\mu$ such that $\langle \eta, \eta' \rangle = 0$ for each $\eta' \in$

$L(\lambda)^+_\mu$ then η is also orthogonal to all elements of $L(\lambda)^+$ because distinct $\mathfrak{gl}(m)$-weight subspaces in $L(\lambda)^+$ are pairwise orthogonal to each other. On the other hand, $L(\lambda) = \mathrm{U}(\mathfrak{n}_-)\, L(\lambda)^+$, where $\mathfrak{n}_-$ is the lower triangular subalgebra in $\mathfrak{gl}(m)$. Therefore, η is orthogonal to $L(\lambda)$ since for any $X \in \mathfrak{n}_-$ and $\eta' \in L(\lambda)$ one has

$$\langle \eta, X\eta' \rangle = \langle X^*\eta, \eta' \rangle = 0$$

because $X^*\eta = 0$. Thus, $\eta = 0$.

We shall maintain the notation $\langle\, , \rangle$ for the restriction of the form to a subspace $L(\lambda)^+_\mu$.

4.2 Quantum minor formulas for generators of Z

Using the homomorphism $\mathrm{Y}(m+n) \to \mathrm{A}(m+n)$ defined by (2.2) we can carry over the quantum minor identities from Section 1 to the matrix $1 + Eu^{-1}$ or, which is more convenient, to $E(u) = u + E$. In accordance with (1.9) and (1.10) we define the polynomials $E^{\,a_1\cdots a_s}_{\,b_1\cdots b_s}(u)$ by the following equivalent formulas

$$E^{\,a_1\cdots a_s}_{\,b_1\cdots b_s}(u) = \sum_{\sigma\in\mathfrak{S}_s} \mathrm{sgn}(\sigma)\, (u+E)_{a_{\sigma(1)}b_1} \cdots (u+E-s+1)_{a_{\sigma(s)}b_s} \tag{4.2}$$

$$= \sum_{\sigma\in\mathfrak{S}_s} \mathrm{sgn}(\sigma)\, (u+E-s+1)_{a_1 b_{\sigma(1)}} \cdots (u+E)_{a_s b_{\sigma(s)}}. \tag{4.3}$$

The polynomial $E^{\,a_1\cdots a_s}_{\,b_1\cdots b_s}(u)$ is skew-symmetric with respect to permutations of the upper indices and of the lower indices. We shall need to evaluate u in $\mathrm{R}(\mathfrak{h})$. To make this operation well-defined let us agree to write the coefficients of this polynomial to the left from powers of u. In the following proposition we use the notation from Section 3.

Proposition 4.1 *For any $i \in \{1, \dots, m\}$ and $a \in \{m+1, \dots, m+n\}$ we have the equalities in $\mathrm{A}'(m+n)$ modulo the ideal* J*:*

$$s_{ia} = (-1)^{i-1} E^{\,1\cdots i}_{\,1\cdots i-1,a}(-h_i), \tag{4.4}$$

$$s_{ai} = E^{\,i+1\cdots m,a}_{\,i\cdots m}(-h_i - i + 1). \tag{4.5}$$

Proof. Let us show first that $E^{\,i+1\cdots m,a}_{\,i\cdots m}(-E_{ii})$ (recall that $h_i = E_{ii} - i + 1$) belongs to the normalizer Norm J of J. By Proposition 1.1 we have the relations

$$[E_{ij}, E^{\,a_1\cdots a_s}_{\,b_1\cdots b_s}(u)] = \sum_{r=1}^{s} \left(\delta_{ja_r} E^{\,a_1\cdots i\cdots a_s}_{\,b_1\cdots b_s}(u) - \delta_{ib_r} E^{\,a_1\cdots a_s}_{\,b_1\cdots j\cdots b_s}(u) \right),$$

where i and j on the right hand side take the rth positions. This implies that $E_{k,k+1}$ commutes with $E^{\, i+1\cdots\, m,a}_{\, i\cdots\, m}(u)$ for $k \in \{1, \dots, m-1\}$, $k \neq i$. Hence,

$$E_{k,k+1} E^{\, i+1\cdots\, m,a}_{\, i\cdots\, m}(-E_{ii}) \in \mathrm{J}.$$

Furthermore,

$$[E_{i,i+1}, E^{\, i+1\cdots\, m,a}_{\, i\cdots\, m}(u)] = E^{\, i,i+2\cdots\, m,a}_{\, i\cdots\, m}(u).$$

We have

$$E^{\, i,i+2\cdots\, m,a}_{\, i\cdots\, m}(u) = (-1)^{m-i} E^{\, i+2\cdots\, m,a,i}_{\, i\cdots\, m}(u),$$

and by (4.3)

$$E^{\, i+2\cdots\, m,a,i}_{\, i\cdots\, m}(u) = \sum_{\sigma} \operatorname{sgn}(\sigma)\, (u+E-m+i)_{i+2,\sigma(i)} \cdots (u+E)_{i,\sigma(m)}. \tag{4.6}$$

However, $\sigma(m)$ take values in $\{i, \dots, m\}$ and so, the element (4.6) belongs to the ideal J for $u = -E_{ii}$. Note that E_{ii} belongs to the normalizer Norm J and therefore $E^{\, i+1\cdots\, m,a}_{\, i\cdots\, m}(-E_{ii})$ does.

To complete the proof of (4.5) we calculate the image $E^{\, i+1\cdots\, m,a}_{\, i\cdots\, m}(-E_{ii})$ under the extremal projection p. By (4.2) we have

$$E^{\, i+1\cdots\, m,a}_{\, i\cdots\, m}(u) = \sum_{\sigma} \operatorname{sgn}(\sigma)\, (u+E)_{\sigma(i+1),i} \cdots (u+E-m+i)_{\sigma(a),m}. \tag{4.7}$$

By the property (3.1) of p we obtain that the image of a summand in (4.7) under p is zero unless $\sigma(i+1) = a$. Further, a summand is zero modulo the ideal J unless

$$\sigma(a) = m, \quad \sigma(m) = m-1, \quad \dots, \quad \sigma(i+2) = i+1.$$

Therefore, the image of (4.7) with $u = -E_{ii}$ equals

$$pE_{ai}\, (h_i - h_{i+1}) \cdots (h_i - h_m),$$

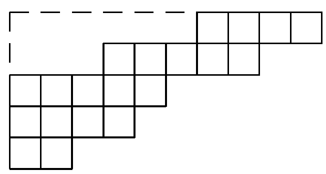
which coincides with s_{ai}. This proves (4.5). The relation (4.4) is proved by a similar argument. $\square$

4.3 Highest vector of the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$

Our aim now is to explicitly construct the highest vector of the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$ and to find its highest weight. This will also allow us to calculate its Drinfeld polynomials.

The action of the Yangian $\mathrm{Y}(n)$ in $L(\lambda)^+_\mu$ is determined by Theorem 3.1. From now on we shall assume that the highest weight λ is a partition, that is, the λ_i are nonnegative integers. This does not lead to a real loss of generality because the formulas and arguments below can be easily adjusted to be valid in the general case. Given a general λ one can add a suitable complex number to all entries of λ to get a partition.

As it follows from the branching rule for the general linear Lie algebras (see [25]) the space $L(\lambda)^+_\mu$ is nonzero only if μ is a partition such that $\mu \subset \lambda$ and each column of the skew diagram λ/μ contains at most n cells. The figure below illustrates the skew diagram for $\lambda = (10, 8, 5, 4, 2)$ and $\mu = (6, 3)$.



Introduce the *row order* on the cells of λ/μ corresponding to reading the diagram by rows from left to right starting from the top row. For a cell $\alpha \in \lambda/\mu$ denote by $r(\alpha)$ the row number of α and by $l(\alpha)$ the (increased) leglength of α which equals 1 plus the number of cells of λ/μ in the column containing α which are below α. Consider the following element of $L(\lambda)$:

$$\zeta = \prod_{\alpha\in\lambda/\mu,\ r(\alpha)\leq m} s_{m+l(\alpha),r(\alpha)}\,\xi, \tag{4.8}$$

where ξ is the highest vector of $L(\lambda)$ and the product is taken in the row order. Using (3.4) we find that $\zeta \in L(\lambda)^+_\mu$. For the above example of λ/μ we have $m = 2$, $n = 3$ and

$$\zeta = (s_{41})^2\,(s_{31})^2\,s_{52}\,s_{42}\,(s_{32})^3\,\xi.$$

Proposition 4.1 allows us to rewrite the definition (4.8) in a different form. Introduce the notation

$$\tau_{ai}(u) = E^{\,i+1\cdots m,a}_{\,i\cdots m}(u), \qquad \tau_{ia}(u) = E^{\,1\cdots i}_{\,1\cdots i-1,a}(u).$$

Then by (4.5) we have

$$\zeta = \prod_{\alpha \in \lambda/\mu,\ r(\alpha) \le m} \tau_{m+l(\alpha),r(\alpha)}(-c(\alpha))\,\xi,$$

where $c(\alpha)$ is the column number of α, and the product is taken in the row order.

Given three integers i, j, k we shall denote by $\mathrm{middle}\{i, j, k\}$ that of the three which is between the two others.

Theorem 4.2 *The vector ζ defined by (4.8) is the highest vector of the $\mathrm{Y}(n)$-module $L(\lambda)^+_\mu$. The highest weight of this module is $(\lambda_1(u), \dots, \lambda_n(u))$ where*

$$\lambda_a(u) = \frac{(u + \nu_a^{(1)})(u + \nu_a^{(2)} - 1) \cdots (u + \nu_a^{(m+1)} - m)}{u(u-1)\cdots(u-m)} \tag{4.9}$$

and

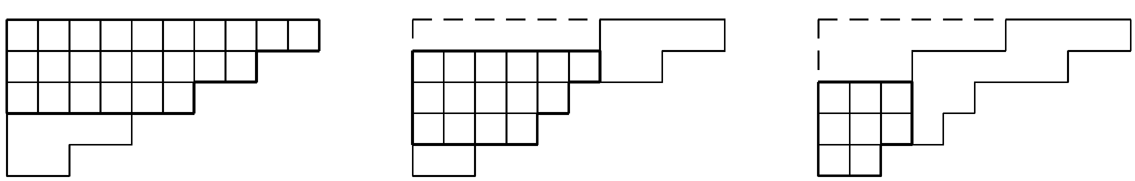
$$\nu_a^{(i)} = \mathrm{middle}\{\mu_{i-1}, \mu_i, \lambda_{a+i-1}\} \tag{4.10}$$

with $\mu_{m+1} = 0$, and μ_0 is considered to be sufficiently large.

Note that for each i the n-tuple $\nu^{(i)} = (\nu_1^{(i)}, \dots, \nu_n^{(i)})$ is a partition which can be obtained from λ/μ as follows. Consider the subdiagram of λ of the form $\lambda^{(i)} = (\lambda_i, \lambda_{i+1}, \dots, \lambda_{i+n-1})$. Replace the rows of $\lambda^{(i)}$ which are longer than μ_{i-1} by μ_{i-1} while those which are shorter than μ_i replace with μ_i and leave the remaining rows unchanged. The resulting partition is $\nu^{(i)}$. For the above example with $\lambda = (10, 8, 5, 4, 2)$ and $\mu = (6, 3)$ we have

$$\nu^{(1)} = (10, 8, 6), \qquad \nu^{(2)} = (6, 5, 4), \qquad \nu^{(3)} = (3, 3, 2),$$

as illustrated:



Proof of Theorem 4.2. Introduce two parameters k and l of the diagram λ/μ as follows. We let k be the row number of the top (non-empty) row of λ/μ if this number is less or equal to m; otherwise set $k = m + 1$. So, if $k \le m$ then we have $\mu_i = \lambda_i$ for $i = 1, \dots, k-1$ and $\mu_k < \lambda_k$. We denote by l the leglength of the cell $\alpha = (k, \mu_k + 1)$

with $k \le m$. We shall prove the following three relations simultaneously by induction on k and l (see (3.9) for the definition of $T_{ab}(u)$):

$$T_{ab}(u)\,\zeta = 0 \tag{4.11}$$

for $m+1 \le a < b \le m+n$,

$$T_{m+a,m+a}(u)\,\zeta = (u+\nu_a^{(1)})(u+\nu_a^{(2)}-1)\cdots(u+\nu_a^{(m+1)}-m)\,\zeta \tag{4.12}$$

for $a = 1,\dots,n$, and

$$s_{m+l,k}\,\zeta = 0, \tag{4.13}$$

if $k \le m$ and $\mu_k = \lambda_{k+l}$.

Suppose that $k = m+1$. Then $\zeta = \xi$ and both (4.11) and (4.12) are immediate from (3.10). Now let $k \le m$ be fixed. We assume that (4.11)–(4.13) hold for all greater values of the parameter k. We proceed by induction on l.

To simplify the notation we shall identify the series $T_{ab}(u)$ with its image under the homomorphism (3.8). That is, we set

$$T_{ab}(u) = E\,{}^{1\cdots m,a}_{1\cdots m,b}(u).$$

We have

$$T_{ab}(u)^* = T_{ba}(u), \tag{4.14}$$

which follows from (4.2) and (4.3). The following relations are derived from Proposition 1.1:

$$T_{ab}(u)\tau_{ci}(v) = \tau_{ci}(v)T_{ab}(u)\,\frac{u-v-i}{u-v-i+1} + \tau_{ai}(v)T_{cb}(u)\,\frac{1}{u-v-i+1}, \tag{4.15}$$

$$T_{ab}(u)\tau_{ci}(v) = \tau_{ci}(v)T_{ab}(u)\,\frac{u-v-i-1}{u-v-i} + T_{cb}(u)\tau_{ai}(v)\,\frac{1}{u-v-i}. \tag{4.16}$$

Using (4.5) replace the first factor $s_{m+l,k}$ in (4.8) by $\tau_{m+l,k}(-\mu_k-1)$ so that the vector ζ will be written in the form $\zeta = \tau_{m+l,k}(-\mu_k-1)\zeta'$. Apply the operator $T_{ab}(u)$ with $a<b$ to ζ. If $b > m+l$ then by (4.15) and (4.5) we have

$$T_{ab}(u)\zeta = \frac{u+\mu_k-k+1}{u+\mu_k-k+2}\,s_{m+l,k}T_{ab}(u)\,\zeta' + \frac{1}{u+\mu_k-k+2}\,s_{ak}T_{m+l,b}(u)\,\zeta'. \tag{4.17}$$

Now (4.11) follows by induction on l and the degree of ζ with respect to $s_{m+l,k}$.

If $b \le m+l$ then $a < m+l$ and by (4.16) and (4.5) we have

$$T_{ab}(u)\zeta = \frac{u+\mu_k-k}{u+\mu_k-k+1}\,s_{m+l,k}T_{ab}(u)\,\zeta' + \frac{1}{u+\mu_k-k+1}\,T_{m+l,b}(u)s_{ak}\,\zeta'. \tag{4.18}$$

However, by (3.5) we have $s_{ak}\,\zeta' = 0$, where we have used the induction hypothesis for (4.13). So, (4.11) follows again by an obvious induction.

Now we use a similar argument to calculate $T_{aa}(u)\,\zeta$. The relation (4.17) with $a = b$ gives

$$T_{aa}(u)\zeta = \frac{u + \mu_k - k + 1}{u + \mu_k - k + 2}\, s_{m+l,k} T_{aa}(u)\, \zeta'$$

for $a > m + l$, and

$$T_{aa}(u)\zeta = s_{m+l,k} T_{aa}(u)\, \zeta'$$

for $a = m + l$. If $a < m + l$ then (4.18) with $a = b$ gives

$$T_{aa}(u)\zeta = \frac{u + \mu_k - k}{u + \mu_k - k + 1}\, s_{m+l,k} T_{aa}(u)\, \zeta'.$$

In all the cases the relation (4.12) follows by an obvious induction.

Let us now prove (4.13). Suppose that $\mu_k = \lambda_{k+l}$ but $\widetilde{\zeta} := s_{m+l,k}\,\zeta \neq 0$. Then $\widetilde{\zeta}$ is a $\mathfrak{gl}(m)$-highest vector of weight $\widetilde{\mu} = \mu - \delta_k$. This is only possible if $\mu_k > \mu_{k+1}$ which will be assumed. We can repeat the previous arguments where the vector ζ is replaced with $\widetilde{\zeta}$ to show that $\widetilde{\zeta}$ is annihilated by $T_{ab}(u)$ with $a < b$ and $\widetilde{\zeta}$ is an eigenvector for the $T_{aa}(u)$. This means that $\widetilde{\zeta}$ is the highest vector of the $\mathrm{Y}(n)$-module $L(\lambda)^+_{\widetilde{\mu}}$. Let us verify that this vector is orthogonal to all elements of $L(\lambda)^+_{\widetilde{\mu}}$. Indeed, by the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}(n)$ (see e.g. [14, Corollary 1.23]), $L(\lambda)^+_{\widetilde{\mu}} = \mathrm{Y}_-\,\widetilde{\zeta}$, where Y_- is the span of monomials in the coefficients of the series $T_{ba}(u)$ with $b > a$. However, due to (4.14) we have for any $\eta \in L(\lambda)^+_{\widetilde{\mu}}$:

$$\langle \widetilde{\zeta}, T_{ba}(u)\eta \rangle = \langle T_{ab}(u)\widetilde{\zeta}, \eta \rangle = 0.$$

It remains to check that $\langle \widetilde{\zeta}, \widetilde{\zeta} \rangle = 0$. Indeed, by (3.7)

$$\langle \widetilde{\zeta}, \widetilde{\zeta} \rangle = \mathrm{const} \cdot \langle \zeta, s_{k,m+l} s_{m+l,k} \zeta \rangle.$$

By (3.11) we have

$$s_{k,m+l} s_{m+l,k} \zeta = (-1)^m T_{m+l,m+l}(-h_k + 1)\, \zeta. \tag{4.19}$$

Note that

$$h_k\, \zeta = (\mu_k - k + 1)\, \zeta = (\lambda_{k+l} - k + 1)\, \zeta.$$

Therefore, (4.19) equals $(-1)^m T_{m+l,m+l}(-\lambda_{k+l}+k)\,\zeta$ which is zero by (4.12). Thus, $\widetilde{\zeta}$ has to be zero which contradicts to our assumption.

Finally, to show that $\zeta \neq 0$ we apply appropriate operators s_{ia} to ζ repeatedly to get the highest vector ξ of $L(\lambda)$ with a nonzero coefficient. Indeed, by (3.11) we have

$$s_{k,m+l}\,\zeta = (-1)^m T_{m+l,m+l}(-\mu_k+k-1)\,\zeta'$$

which equals

$$-(\mu_k-\nu_l^{(1)}-k+1)\cdots(\mu_k-\nu_l^{(m+1)}-k+m+1)\,\zeta' \tag{4.20}$$

by (4.12). Here the numbers $\nu_l^{(i)}$ are defined by (4.10) with

$$\mu=(\lambda_1,\dots,\lambda_{k-1},\mu_k+1,\mu_{k+1},\dots,\mu_m).$$

So, the coefficient at ζ' in (4.20) is nonzero. □

Theorem 4.2 implies the following corollary. As usual, by the *content* of a cell $\alpha=(i,j)\in\lambda/\mu$ we mean the number $j-i$.

Corollary 4.3 *The Drinfeld polynomials for the* $\mathrm{Y}(n)$*-module* $L(\lambda)^+_\mu$ *are given by*

$$P_a(u)=\prod_c (u+c), \qquad a=1,\dots,n-1, \tag{4.21}$$

where c runs over the contents of the top cells of columns of height a in the diagram λ/μ.

Proof. By the definition (4.1) and the formula (4.9) we have

$$P_a(u)=\prod_{k=1}^{m+1}(u+\nu_{a+1}^{(k)}-k+1)\cdots(u+\nu_a^{(k)}-k),$$

where the kth factor is assumed to be equal to 1 if $\nu_a^{(k)}=\nu_{a+1}^{(k)}$. Now (4.10) implies that this product coincides with (4.21). □

If $\lambda=(10,8,5,4,2)$ and $\mu=(6,3)$ (see the example above) then we have

$$P_1(u)=(u+4)(u+8)(u+9), \qquad P_2(u)=u(u+3)(u+6)(u+7).$$

The corollary shows that the $\mathrm{Y}(\mathfrak{sl}(n))$-module $L(\lambda)^+_\mu$ is isomorphic to the corresponding elementary representation of $\mathrm{Y}(\mathfrak{sl}(n))$ which was studied in [17].

References

[1] R. M. Asherova, Yu. F. Smirnov and V. N. Tolstoy, *Projection operators for simple Lie groups*, Theor. Math. Phys. **8** (1971), 813–825.

[2] V. Chari and A. Pressley, *Yangians and R-matrices*, L'Enseign. Math. **36** (1990), 267–302.

[3] I. V. Cherednik, *A new interpretation of Gelfand–Tzetlin bases*, Duke Math. J. **54** (1987), 563–577.

[4] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars, Paris, 1974.

[5] V. G. Drinfeld, *Hopf algebras and the quantum Yang–Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.

[6] V. G. Drinfeld, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** (1988), 212–216.

[7] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), 218–348.

[8] I. M. Gelfand and V. S. Retakh, *Determinants of matrices over noncommutative rings*, Funct. Anal. Appl. **25** (1991), 91–102.

[9] M. D. Gould and N. I. Stoilova, *Casimir invariants and characteristic identities for* $\mathfrak{gl}(\infty)$, J. Math. Phys. **38** (1997), 4783–4793.

[10] D. Krob and B. Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm. Math. Phys. **169** (1995), 1–23.

[11] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, *Yang–Baxter equation and representation theory*, Lett. Math. Phys. **5** (1981), 393–403.

[12] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method: recent developments*, in 'Integrable Quantum Field Theories', Lecture Notes in Phys. **151** Springer, Berlin-Heidelberg, 1982, pp. 61–119.

[13] A. I. Molev, *Finite-dimensional irreducible representations of twisted Yangians*, J. Math. Phys. **39** (1998), 5559–5600.

[14] A. Molev, M. Nazarov and G. Olshanski, *Yangians and classical Lie algebras*, Russian Math. Surveys **51**:2 (1996), 205–282.

[15] A. Molev and G. Olshanski, *Centralizer construction for twisted Yangians*, Preprint CMA 065-97, Austral. Nat. University, Canberra; q-alg/9712050.

[16] M. Nazarov and V. Tarasov, *Yangians and Gelfand–Zetlin bases*, Publ. RIMS, Kyoto Univ. **30** (1994), 459–478.

[17] M. Nazarov and V. Tarasov, *Representations of Yangians with Gelfand–Zetlin bases*, J. Reine Angew. Math. **496** (1998), 181–212.

[18] A. Okounkov, *Quantum immanants and higher Capelli identities*, Transformation Groups **1** (1996), 99–126.

[19] A. Okounkov and G. Olshanski, *Shifted Schur functions*, St. Petersburg Math. J. **9** (1998), 239–300.

[20] G. I. Olshanski, *Extension of the algebra $U(g)$ for infinite-dimensional classical Lie algebras g, and the Yangians $Y(gl(m))$*. Soviet Math. Dokl. **36** (1988), no. 3, 569–573.

[21] G. I. Olshanski, *Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians*, in 'Topics in Representation Theory' (A. A. Kirillov, Ed.), Advances in Soviet Math. **2**, AMS, Providence RI, 1991, pp. 1–66.

[22] G. I. Olshanski, *Twisted Yangians and infinite-dimensional classical Lie algebras*, in 'Quantum Groups' (P. P. Kulish, Ed.), Lecture Notes in Math. **1510**, Springer, Berlin-Heidelberg, 1992, pp. 103–120.

[23] L. A. Takhtajan and L. D. Faddeev, *Quantum inverse scattering method and the Heisenberg XYZ-model*, Russian Math. Surv. **34** (1979), no. 5, 11–68.

[24] H. Weyl, *Classical Groups, their Invariants and Representations*, Princeton Univ. Press, Princeton NJ, 1946.

[25] D. P. Želobenko, *Compact Lie groups and their representations*, Transl. of Math. Monographs **40** AMS, Providence RI 1973.

[26] D. P. Zhelobenko, *S-algebras and Verma modules over reductive Lie algebras*, Soviet. Math. Dokl. **28** (1983), 696–700.

[27] D. P. Zhelobenko, *Z-algebras over reductive Lie algebras*, Soviet. Math. Dokl. **28** (1983), 777–781.

[28] D. P. Zhelobenko, *An introduction to the theory of S-algebras over reductive Lie algebras*, in ‘Representations of Lie groups and Related Topics’ (A. M. Vershik, D. P. Zhelobenko, Eds.), Adv. Studies in Contemp. Math. **7**, Gordon and Breach Science Publishers, New York, 1990, pp. 155–221.